# The Complexity of the Hamilton Cycle Problem in Hypergraphs of High Minimum Codegree*

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## Abstract

We consider the complexity of the Hamilton cycle decision problem when restricted to $k$-uniform hypergraphs $H$ of high minimum codegree $\delta(H)$. We show that for tight Hamilton cycles this problem is NP-hard even when restricted to $k$-uniform hypergraphs $H$ with $\delta(H) \geq \frac{n}{2} - C$, where $n$ is the order of $H$ and $C$ is a constant which depends only on $k$. This answers a question raised by Karpiński, Ruciński and Szymańska. Additionally we give a polynomial-time algorithm which, for a sufficiently small constant $\varepsilon > 0$, determines whether or not a 4-uniform hypergraph $H$ on $n$ vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ contains a Hamilton 2-cycle. This demonstrates that some looser Hamilton cycles exhibit interestingly different behaviour compared to tight Hamilton cycles. A key part of the proof is a precise characterisation of all 4-uniform hypergraphs $H$ on $n$ vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ which do not contain a Hamilton 2-cycle; this may be of independent interest. As an additional corollary of this characterisation, we obtain an exact Dirac-type bound for the existence of a Hamilton 2-cycle in a large 4-uniform hypergraph.

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## 1 Introduction

The study of Hamilton cycles in graphs has been a topic of great significance in graph theory, and continues to be well-studied. For example, the Hamilton cycle decision problem (given a graph, determine whether it contains a Hamilton cycle) was one of Karp's celebrated 21 NP-complete problems [9], whilst one very well-known classic result is Dirac's theorem [4], which states that any graph on $n \geq 3$ vertices with minimum degree at least $\frac{n}{2}$ contains a Hamilton cycle.

The problem of generalising these results to the hypergraph setting has been a highly-active area of research over the past several years (see, for example, the recent surveys by Kühn and Osthus [15], Rödl and Ruciński [16] and Zhao [21]). To describe these developments we require the following standard definitions. A *$k$-uniform hypergraph*, or *$k$-graph* $H$ consists of a set of vertices $V(H)$ and a set of edges $E(H)$, where each edge consists of $k$ vertices. So a 2-graph is a (simple) graph. We say that a $k$-graph $C$ is an *$\ell$-cycle* if its vertices can be cyclically ordered in such a way that each edge of $C$ consists of $k$ consecutive vertices,

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and so that each edge intersects the subsequent edge in $\ell$ vertices. This naturally generalises the notion of a cycle in a graph, and is the most commonly-used definition of a hypergraph cycle. However, various other definitions have also been considered, such as a Berge cycle [1]. Note in particular that each edge of an $\ell$-cycle $k$-graph $C$ has $k-\ell$ vertices which were not contained in the previous edge, so the number of vertices of $C$ must be divisible by $k-\ell$. We say that a $k$-graph $H$ on $n$ vertices contains a *Hamilton $\ell$-cycle* if it contains an $n$-vertex $\ell$-cycle as a subgraph; as before, this is only possible if $k-\ell$ divides $n$. We refer to $(k-1)$-cycles as *tight cycles*, and in the same way refer to *tight Hamilton cycles.* Given a $k$-graph $H$ and a set $S \subseteq V(H)$, the *degree* of $S$, denoted $\deg_H(S)$ (or $\deg(S)$ when $H$ is clear from the context), is the the number of edges of $H$ which contain $S$ as a subset. The *minimum codegree* of $H$, denoted $\delta(H)$, is the minimum of $\deg(S)$ taken over all sets of $k-1$ vertices of $H$, and the *maximum codegree* of $H$, denoted $\Delta(H)$, is the maximum of $\deg(S)$ taken over all sets of $k-1$ vertices of $H$. In the graph case the maximum and minimum codegree are simply the maximum and minimum degree respectively.

An elementary reduction from the graph case demonstrates that for any $k \geq 3$ and $1 \leq \ell \leq k$ the Hamilton $\ell$-cycle decision problem (given a $k$-graph $H$, determine whether it contains a Hamilton $\ell$-cycle) is also NP-complete. For this reason, many authors have asked for conditions on $H$ which render this problem tractable, or which guarantee the existence of a Hamilton $\ell$-cycle in $H$. In particular, since a Hamilton cycle in $H$ cannot exist if $H$ has an isolated vertex, it is natural to study minimum degree conditions on $H$.

## 1.1 Dirac-Type Results

The following theorem, whose various cases were proved by Rödl, Ruciński and Szemerédi [17, 18], Kühn and Osthus [14], Keevash, Kühn, Osthus and Mycroft [12], Hàn and Schacht [6], and Kühn, Osthus and Mycroft [13], is an approximate hypergraph analogue of Dirac's theorem; for any $k$ and $\ell$ it gives the asymptotically best-possible minimum codegree condition which guarantees the existence of a Hamilton $\ell$-cycle in a $k$-graph.

▶ **Theorem 1** ([6, 12, 13, 14, 17, 18]). *For any $k \geq 3$, $1 \leq \ell \leq k-1$ and $\eta > 0$, there exists $n_0$ such that if $n \geq n_0$ is divisible by $k-\ell$ and $H$ is a $k$-graph on $n$ vertices with*

$$\delta(H) \geq \begin{cases} \left(\frac{1}{2}+\eta\right)n & \text{if } k-\ell \text{ divides } k, \\ \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)}+\eta\right)n & \text{otherwise,} \end{cases}$$

*then $H$ contains a Hamilton $\ell$-cycle.*

Simple constructions show that for any $k$ and $\ell$ this minimum codegree condition is best possible up to the $\eta n$ error term. More recently the exact threshold (for large $n$) has been determined in some cases: for $k=3, \ell=2$ by Rödl, Ruciński and Szemerédi [19], for $k=3, \ell=1$ by Czygrinow and Molla [2], and for $k \geq 3$ and $\ell < k/2$ by Han and Zhao [8]. As part of our work on the question of tractability (described in more detail in the next section), we successfully characterised all 4-graphs $H$ with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ which do not contain a Hamilton cycle. As a straightforward consequence of this, we add to the aforementioned results the exact Dirac-type statement for the previously-open case $k=4, \ell=2$.

▶ **Theorem 2.** *There exists $n_0$ such that if $n \geq n_0$ is even and $H$ is a 4-graph on $n$ vertices with*

$$\delta(H) \geq \begin{cases} \frac{n}{2}-2 & \text{if } n \text{ is divisible by } 8, \\ \frac{n}{2}-1 & \text{otherwise,} \end{cases}$$

*then $H$ contains a Hamilton 2-cycle. Moreover, this condition is best-possible for any even $n \geq n_0$.*

## 1.2 Tractability of the Restricted Hamilton Cycle Decision Problem

We now turn to the primary focus of this paper: minimum degree conditions which render the Hamilton cycle decision problem tractable. In the graph case, Dahlhaus, Hajnal and Karpiński [3] showed that for any fixed $\varepsilon > 0$ this problem remains NP-complete when restricted to graphs with minimum degree at least $(1-\varepsilon)\frac{n}{2}$. More recently, Karpiński, Ruciński and Szymańska [10] showed that for any $k \geq 3$ and any fixed $\varepsilon > 0$ the tight Hamilton cycle decision problem remains NP-complete when restricted to $k$-graphs with minimum codegree $(1-\varepsilon)\frac{n}{k}$. They noted that, combined with Theorem 1, this left a 'hardness gap' of $[\frac{n}{k}, \frac{n}{2}]$ for which the hardness of the problem remained unknown. We answer this question with the following theorem.

▶ **Theorem 3.** *For any $k \geq 3$ there exists $C$ such that the tight Hamilton cycle decision problem remains NP-complete when restricted to $k$-graphs $H$ with $\delta(H) \geq \frac{n}{2} - C$ (where $n = |V(H)|$).*

Assuming that P $\neq$ NP, Theorems 1 and 3 together imply that the minimum codegree threshold at which the tight Hamilton cycle decision problem becomes tractable is asymptotically equal to the minimum codegree threshold for the existence of a tight Hamilton cycle, mirroring the situation in the graph case. Interestingly, we can demonstrate that the Hamilton 2-cycle problem exhibits significantly different behaviour; our next theorem shows that there is a linear-size gap between the threshold at which the problem becomes tractable and at which the existence of a cycle is guaranteed.

▶ **Theorem 4.** *There exist a constant $\varepsilon > 0$ and an algorithm which, given a 4-graph $H$ on $n$ vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$, determines in time $O(n^{25})$ whether $H$ contains a Hamilton 2-cycle.*

A slight adaptation of the argument of Karpiński, Ruciński and Szymańska [10] mentioned above shows that for any fixed $\varepsilon > 0$ the Hamilton 2-cycle problem remains NP-complete when restricted to 4-graphs with minimum codegree at least $(1-\varepsilon)\frac{n}{4}$.

A key result in our proof of Theorem 4, which may be of independent interest, is Theorem 6, which (for sufficiently small $\varepsilon$ and large $n$) precisely characterises all 4-graphs on $n$ vertices which satisfy $\delta(H) \geq \frac{n}{2} - \varepsilon n$ but which do not contain a Hamilton 2-cycle. We prove this result using recently developed techniques of extremal graph theory, in particular the so-called 'absorbing method' of Rödl, Ruciński and Szemerédi [17]. Establishing this characterisation is the principal difficulty in the proof of Theorem 4, as then the algorithm for Theorem 4 simply checks whether this characterisation is satisfied. Likewise, Theorem 2 follows from Theorem 6 by a case analysis.

## 1.3 Discussion

In the light of Theorem 4, it would be very interesting to know which other values of $k$ and $\ell$ also have the property that there is a linear-size gap between the minimum codegree threshold which renders the $k$-graph Hamilton $\ell$-cycle problem tractable and the minimum codegree threshold under which the problem becomes trivial. Theorem 3 shows that this is not the case when $\ell = k-1$, whilst a slight adaptation to the arguments of Karpiński, Ruciński and Szymańska [10] demonstrates that this is also not true if $k-\ell$ does not divide $k$ (in which case the lower degree threshold of Theorem 1 applies); all other cases remain open.

We also note that Theorem 3 demonstrates an interesting difference between the perfect matching problem and tight Hamilton cycle problem in $k$-graphs. Indeed, while the unrestricted versions of both problems are NP-complete, Keevash, Knox and Mycroft [11] and Han [7] showed that the perfect matching problem can be solved in polynomial time in $k$-graphs $H$ with $\delta(H) \geq n/k$; complementing a previous result of Szymanska [20], who showed that for any $\varepsilon > 0$ the problem remains NP-complete under the restriction $\delta(H) \geq (\frac{1}{k} - \varepsilon)n$. So, assuming P $\neq$ NP, for any $\frac{1}{k} \leq \alpha < \frac{1}{2}$ the two problems lie in distinct complexity classes when restricted to $k$-graphs with minimum codegree $\delta(H) \geq \alpha n$.

Finally, whilst the constant $\varepsilon$ in Theorem 4 is quite small, we conjecture that Theorem 6 (the characterisation of 4-graphs $H$ with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ and no Hamilton 2-cycle) is in fact valid under the weaker condition that $\delta(H) > \frac{n}{3}$. If true, this would imply that Theorem 4 would also hold under this weaker codegree assumption.

## 1.4 Notation

Given a set $V$, we write $\binom{V}{k}$ for the set of subsets of $V$ of size $k$. Also, we write $x \ll y$ ("$x$ is sufficiently smaller than $y$") to mean that for any $y > 0$ there exists $x_0 > 0$ such that for any $x \leq x_0$ the subsequent statement holds. Similar statements with more variables are defined accordingly.

# 2 Hamilton 2-Cycles

In this section we outline the proof of Theorem 4. The key to the proof is Theorem 6, which precisely characterises all large 4-graphs $H$ with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ which do not contain a Hamilton 2-cycle. This is presented in Section 2.1. Having established this characterisation, it is fairly straightforward to exhibit a polynomial-time algorithm which tests whether a 4-graph has this property, as shown in Section 2.2. Instead, the difficult part of the proof is to prove Theorem 6; we outline how this is done in Section 2.3. Finally, in Section 2.4 we present the short deduction of Theorem 2 from Theorem 6.

## 2.1 A Characterisation of Dense 4-graphs with no Hamilton 2-Cycle

For 4-graphs $H$, our characterisation considers partitions of $V(H)$ into two parts $A$ and $B$. Whenever we refer to, for example, 'a partition $(A, B)$ of $V(H)$', this should be interpreted as meaning a partition of $V(H)$ into two non-empty parts $A$ and $B$. Given such a partition of $V(H)$, we say that an edge $e \in E(H)$ is *odd* if $|e \cap A|$ is odd, and *even* if $|e \cap A|$ is even. We write $H_{\text{even}}$ for the subgraph of $H$ consisting only of even edges of $H$, and similarly write $H_{\text{odd}}$ for the subgraph of $H$ consisting only of odd edges of $H$. Also, we say that a pair $\{x, y\}$ of distinct vertices of $H$ is a *split* pair if $x \in A$ and $y \in B$ or vice versa, and that $\{x, y\}$ is an *equal* pair if $x, y \in A$ or $x, y \in B$.

We define an $\ell$-path in a $k$-graph analogously to an $\ell$-cycle: a $k$-graph is an *$\ell$-path* if its vertices can be linearly ordered $v_1, \ldots, v_n$ such that every edge consists of $k$ consecutive vertices and successive edges intersect in precisely $\ell$ vertices. As for cycles we refer to $(k-1)$-paths as *tight paths*. The *length* of an $\ell$-path is the number of edges. Given a 4-graph $H$, we define the *total 2-pathlength* of $H$ to be the maximum sum of lengths of vertex-disjoint 2-paths in $H$. For example, $H$ having total 2-pathlength 3 could be achieved by 3 disjoint edges (i.e. 2-paths of length 1) in $H$, or a 2-path of length 3 in $H$, or two vertex-disjoint 2-paths in $H$, one of length 1 and one of length 2. Using these definitions we can now give the central definition of our characterisation.

▶ **Definition 5.** Let $H$ be a 4-graph on $n$ vertices, where $n$ is even. We say that a partition $(A, B)$ of $V(H)$ is *even-good* if at least one of the following statements holds:

**(i)** $|A|$ is even or $|A| = |B|$.
**(ii)** $H$ contains odd edges $e$ and $e'$ such that either $e \cap e' = \emptyset$ or $e \cap e'$ is a split pair.
**(iii)** $|A| = |B| + 2$ and $H$ contains odd edges $e$ and $e'$ with $e \cap e' \in \binom{A}{2}$.
**(iv)** $|B| = |A| + 2$ and $H$ contains odd edges $e$ and $e'$ with $e \cap e' \in \binom{B}{2}$.

Now let $m \in \{0, 2, 4, 6\}$ and $d \in \{0, 2\}$ be such that $m \equiv n \mod 8$ and $d \equiv |A| - |B| \mod 4$. Then we say that $(A, B)$ is *odd-good* if at least one of the following statements holds.

**(v)** $(m, d) \in \{(0, 0), (4, 2)\}$.
**(vi)** $(m, d) \in \{(2, 2), (6, 0)\}$ and $H$ contains an even edge.
**(vii)** $(m, d) \in \{(4, 0), (0, 2)\}$ and $H_{even}$ has total 2-pathlength at least two.
**(viii)** $(m, d) \in \{(6, 2), (2, 0)\}$ and either there is an edge $e \in E(H)$ with $|e \cap A| = |e \cap B| = 2$ or $H_{even}$ has total 2-pathlength at least three.

A key observation is that if $(A, B)$ is a partition of $V(H)$ which is not even-good, then there exists a set $X$ of at most four vertices of $H$ such that every odd edge of $H$ intersects $X$. Indeed, if $H$ contains an odd edge $e$, then we may take $X = e$, and otherwise we may take $X = \emptyset$. Similarly, by choosing $X$ to be the vertices of at most two disjoint even edges, or of a 2-path of length two in $H_{\text{even}}$, we find that if $(A, B)$ is a partition of $V(H)$ which is not odd-good, then there exists a set $X$ of at most 8 vertices of $H$ such that every even edge of $H$ intersects $X$.

We now give our characterisation of 4-graphs of high minimum codegree with no Hamilton 2-cycle. Recall for this that any 2-cycle 4-graph has an even number of vertices.

▶ **Theorem 6.** *There exist $\varepsilon, n_0 > 0$ such that the following statement holds for any even $n \geq n_0$. Let $H$ be a 4-graph on $n$ vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. Then $H$ admits a Hamilton 2-cycle if and only if every partition $(A, B)$ of $V(H)$ is both even-good and odd-good.*

## 2.2 The Algorithm

Our polynomial-time algorithm for determining the existence of a Hamilton 2-cycle in a 4-graph of high codegree makes use of a special case of a result of Keevash, Knox and Mycroft [11]. This result allows us to efficiently list all partitions $(A, B)$ of $V(H)$ with no odd edges, or all partitions with no even edges.

▶ **Lemma 7** ([11], special case of Lemma 2.2)**.** *Let $H$ be a 4-graph on $n$ vertices with $\delta(H) > \frac{n}{3}$, and let $x \in \{\text{even}, \text{odd}\}$. Then there are at most 64 partitions $(A, B)$ of $V(H)$ for which no edge of $H$ has parity $x$ with respect to $(A, B)$. Moreover, there exists an algorithm* ListPartitions$(H, x)$ *with running time $O(n^5)$ which, given $H$ and $x$, returns all such partitions.*

We now present an algorithm, Procedure GoodPartition$(H, x)$, which determines whether or not there exists an even-good/odd-good partition $(A, B)$ for a 4-graph $H$. Note that, given a 4-graph $H$ and a partition $(A, B)$ of $V(H)$, the truth of the statements '$(A, B)$ is odd-good' and '$(A, B)$ is even-good' depend only on the values of $n$ and $|A|$ and whether or not $H_{\text{odd}}$ or $H_{\text{even}}$ contain certain subgraphs with at most 12 vertices. It follows that the validity of these statements (and therefore the condition of the 'if' statement in Procedure GoodPartition) can be tested in time $O(n^{12})$.

▶ **Proposition 8.** *Let $H$ be a 4-graph on $n$ vertices with $\delta(H) > \frac{n}{3}$, where $n$ is even, and fix a parity $x \in \{\text{even}, \text{odd}\}$. Then Procedure GoodPartition$(H, x)$ will correctly determine whether there exists a partition $(A, B)$ of $V(H)$ which is not $x$-good, with running time $O(n^{25})$.*

**Procedure** GoodPartition$(H, x)$

**Data:** A 4-graph $H$ with vertex set $V$ and a parity $x \in \{\text{even}, \text{odd}\}$.
**Result:** Determines if there is a partition $(A, B)$ of $V$ which is not $x$-good.

**for** *each set* $X \subseteq V(H)$ *with* $|X| = 8$ **do**
  Let $V' = V \setminus X$ and $H' = H[V']$.
  Run Procedure ListPartitions$(H', x)$ to obtain all partitions $(A', B')$ of $V'$ with no edges not of parity $x$.
  **for** *each such partition* $(A', B')$ **do**
    **for** *each partition* $(A, B)$ *of* $V$ *with* $A' \subseteq A$ *and* $B' \subseteq B$ **do**
      **if** $(A, B)$ *is not* $x$*-good* **then**
        State '$(A, B)$ is not $x$-good', and terminate.
State 'Every partition is $x$-good', and terminate.

**Proof.** We first establish correctness of the algorithm; for this, fix $H$ and $x$ as in the proposition statement. Clearly, if every partition $(A, B)$ of $V := V(H)$ is $x$-good, then GoodPartition$(H, x)$ will output this fact. So suppose that some partition $(A, B)$ of $V$ is not $x$-good. As noted following Definition 5, we may then choose a set $X$ of at most 8 vertices of $H$ which is intersected by every edge of $H$ which does not have parity $x$. This means that when GoodPartition$(H, x)$ considers this set $X$, ListPartitions will return the partition $(A', B')$ where $A' = A \setminus X$ and $B' = B \setminus X$, and at this point GoodPartition$(H, \text{even})$ will return that $(A, B)$ is not $x$-good, as required.

Finally we consider the running time of the algorithm. For this note that there are $\binom{n}{8}$ choices for $X$ in the outside 'for loop', and for each of these Procedure ListPartitions$(H', x)$ runs in time $O(n^5)$. The inside 'for loops' then range over sets of size at most 64 (by Lemma 7) and $2^8 = 256$ respectively. Finally, as noted above we may test whether a partition $(A, B)$ is $x$-good in time $O(n^{12})$; together these bounds combine to give the claimed running time. ◀

**Proof of Theorem 4.** Let $n_0$ be sufficiently large and $\varepsilon > 0$ sufficiently small for Theorem 6 to apply. Given a 4-graph $H$ on $n$ vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$, we apply the following algorithm. Firstly, if $n$ is odd, then there can be no Hamilton 2-cycle in $H$, so we output this fact and terminate. Secondly, if $n < n_0$, then we use a brute-force approach, testing each of the at most $n_0!$ orderings of $V(H)$ in turn to determine whether it yields a Hamilton 2-cycle in $H$. We then output the appropriate answer and terminate. Finally, if $n \geq n_0$ is even, then we first run Procedure GoodPartition$(H, \text{even})$, and then run Procedure GoodPartition$(H, \text{odd})$. If either of these procedures yields a partition $(A, B)$ of $V(H)$ which is not even-good or which is not odd-good, then we return that there is no Hamilton 2-cycle in $H$, otherwise we return that there is such a cycle. Note that in the first two cases this algorithm runs in constant time, whilst in the final case it runs in time $O(n^{25})$ by Proposition 8. Moreover Theorem 6 ensures that this algorithm will always output the correct answer. ◀

## 2.3 Proof of Theorem 6

We begin by establishing the forward implication of Theorem 6, expressed in the following proposition. In fact, the minimum codegree condition on $H$ is not required for this direction.

▶ **Proposition 9.** *If $H$ is a 4-graph which contains a Hamilton 2-cycle, then every partition $(A, B)$ of $V(H)$ is both even-good and odd-good.*

**Proof.** Let $n$ be the order of $H$, let $C = (v_1, v_2, \ldots, v_n)$ be a Hamilton 2-cycle in $H$ and let $(A, B)$ be a partition of $V(H)$. Write $P_i = \{v_{2i-1}, v_{2i}\}$ for each $1 \leq i \leq \frac{n}{2}$, so the edges of $C$ are $e_i := P_i \cup P_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ (with addition taken modulo $\frac{n}{2}$). The key observation is that $e_i$ is even if $P_i$ and $P_{i+1}$ are both split pairs or both equal pairs, and odd otherwise.

We first show that $(A, B)$ is even-good. This holds by (ii) if $H$ contains two disjoint odd edges, so we may assume without loss of generality that all edges of $H$ other than $e_1$ and $e_{n/2}$ are even. It follows that the pairs $P_2, P_3, \ldots, P_{n/2}$ are either all split pairs or all equal pairs. In the former case, if $P_1$ is a split pair then $|A| = |B|$, so (i) holds, whilst if $P_1 \subseteq A$ then (iii) holds, and if $P_1 \subseteq B$ then (iv) holds. In the latter case, if $P_1$ is an equal pair then $|A|$ is even, so (i) holds, whilst if $P_1$ is a split pair then (ii) holds. So in all cases we find that $(A, B)$ is even-good.

To show that $(A, B)$ is odd-good, suppose first that 4 does not divide $n$, and note that by our key observation the number of even edges in $C$ must then be odd. If $C$ contains three or more even edges or an edge with precisely two vertices in $A$, then $(A, B)$ is odd-good by (vi) and (viii), so we may assume without loss of generality that $e_{n/2}$ is the unique even edge in $C$ and that $e_{n/2} \subseteq A$ or $e_{n/2} \subseteq B$. It follows that $P_1, P_3, \ldots, P_{n/2}$ are equal pairs and the remaining pairs are split, so $|A| - |B| \equiv 2\lceil \frac{n}{4} \rceil \mod 4$. We must therefore have $(m, d) \in \{(2, 2), (6, 0)\}$, and $(A, B)$ is odd-good by (vi). On the other hand, if 4 divides $n$, then by our key observation the number of even edges in $C$ is even. If this number is at least two then $(A, B)$ is odd-good by (v) and (vii). If instead every edge of $C$ is odd, then exactly $\frac{n}{4}$ of the pairs $P_i$ are equal pairs, so $|A| - |B| \equiv \frac{n}{2} \mod 4$, and $C$ is odd-good by (v). ◀

To prove Theorem 6 it therefore suffices to prove the backwards implication. Our approach for this is motivated by the observation that if $H$ is a 4-graph and $(A, B)$ is a partition of $V(H)$ which is not odd-good, then $H$ must have very few even edges. Likewise, if $(A, B)$ is not even-good, then $H$ has very few odd edges. We therefore consider three cases for $H$: two 'near-extremal' cases, in which $V(H)$ admits a partition $(A, B)$ with few even edges or with few odd edges, and a 'non-extremal' case, in which there is no such partition. In the 'non-extremal case' we proceed by the so-called 'absorbing' method, introduced by Rödl, Ruciński and Szemerédi [19], in which we rely heavily on the fact that $H$ is not 'near-extremal'. On the other hand, in the 'near-extremal' cases we have significant information about the structure of $H$ (specifically that there is a partition of $V(H)$ with few even/odd edges). Making essential use of this structural information, we proceed by *ad hoc* methods to construct a Hamilton 2-cycle in $H$.

The following definition formalises our two notions of 'near-extremal'.

▶ **Definition 10.** Let $c_1, c_2 > 0$ and let $H$ be a 4-graph on $n$ vertices.

**(a)** We say that $H$ is $c_1$-*even-extremal* if there exists a partition $(A, B)$ of $V(H)$ such that $(\frac{1}{2} - c_1)n \leq |A| \leq (\frac{1}{2} + c_1)n$ and $H$ contains at most $c_1\binom{n}{4}$ odd edges.

**(b)** We say that $H$ is $c_2$-*odd-extremal*, if there exists a partition $(A, B)$ of $V(H)$ such that $(\frac{1}{2} - c_2)n \leq |A| \leq (\frac{1}{2} + c_2)n$ and $H$ contains at most $c_2\binom{n}{4}$ even edges.

### 2.3.1 Non-Extremal 4-Graphs

As described above, in the case when $H$ is not near-extremal, we proceed by the 'absorbing' method of Rödl, Ruciński and Szemerédi [19]. To do this we establish three key lemmas. The first of these is a 'connecting lemma', which shows that since $H$ is not even-extremal, we can find a constant-length 2-path connecting any two disjoint pairs of vertices. For this, we say that the *ends* of a 2-path 4-graph $(v_1, \ldots, v_n)$ are the pairs $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$.

▶ **Lemma 11** (Connecting lemma). *Suppose that $\frac{1}{n} \ll \varepsilon \ll c$ and that $H$ is a 4-graph on $n$ vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ which is not $c$-even-extremal. Then for every two disjoint pairs $\{a_1, a_2\}, \{b_1, b_2\} \in \binom{V}{2}$ there is a 2-path of length at most 3 whose ends are $\{a_1, a_2\}$ and $\{b_1, b_2\}$.*

Loosely speaking, our proof of Lemma 11 supposes that we have pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$ for which no such 2-path exists. It follows that there is no pair $\{x, y\} \in \binom{V(H)}{2}$ for which $\{a_1, a_2, x, y\}$ and $\{b_1, b_2, x, y\}$ are both edges of $H$. Combined with the minimum codegree condition of $H$ this yields significant structural information on $H$, which we use to deduce that $H$ must be $c$-even-extremal and so prove the lemma.

The second key lemma is an 'absorbing lemma', which shows that since $H$ is neither even-extremal nor odd-extremal, we can find a short 2-path in $H$ which can 'absorb' most small collections of pairs of $H$.

▶ **Lemma 12** (Absorbing lemma). *Suppose that $\frac{1}{n} \ll \varepsilon \ll \rho \ll \beta \ll \lambda \ll c, \mu$. Let $H$ be a 4-graph on $n$ vertices with $\delta(H) \geq \frac{n}{2} - \varepsilon n$ which is neither $c$-even-extremal nor $c$-odd-extremal. Then there is a 2-path $P$ in $H$ and a graph $G$ on $V(H)$ with the following properties.*

**(i)** *$P$ has at most $\mu n$ vertices.*

**(ii)** *Every vertex of $V(H) \setminus V(P)$ lies in at least $(1 - \lambda)n$ edges of $G$.*

**(iii)** *For any $q \leq \rho n$ and any $q$ disjoint edges $e_1, \dots, e_q$ of $G$ which do not intersect $P$ there is a 2-path $P^*$ in $H$ with the same ends as $P$ such that $V(P^*) = V(P) \cup \bigcup_{j=1}^{q} e_j$.*

Loosely speaking, to prove Lemma 12, we first show that provided $H$ is not $c$-odd-extremal, for almost every pair $\{x, y\} \in \binom{V}{2}$ there are many 2-paths $Q$ of length 3 which can 'absorb' $\{x, y\}$, in the sense that there is a 2-path $Q^*$ with vertex set $V(Q) \cup \{x, y\}$ and with the same ends as $Q$. We take $G$ to be the graph of such pairs. We then randomly select a linear number of 2-paths of length 3 and use Lemma 11 to connect these 2-paths into a single short 2-path $P$ (this is where we require that $H$ is not $c$-even-extremal). Next we extend $P$ to include the small number of vertices which lie in fewer than $(1 - \lambda)n$ edges of $G$, so that (ii) holds. Finally, we show that given any set of edges $e_1, \dots, e_q$ of $G$ as in (iii), we can match these edges to the randomly chosen paths $Q$, and absorb each edge into the corresponding path to obtain $P^*$.

Our final key lemma is a 'path cover lemma', which states that we can cover almost all vertices of $H$ by a constant number of vertex-disjoint 2-paths. In fact, we do not actually need the requirement that $H$ is not near-extremal, and can simply cite a result of Kühn, Mycroft and Osthus [13].

▶ **Lemma 13** (Path cover lemma [13]). *Suppose that $\frac{1}{n} \ll \frac{1}{D} \ll \gamma \ll \eta$ and that $H$ is a 4-graph on $n$ vertices with $\delta(H) \geq (\frac{1}{4} + \eta)n$. Then $H$ contains a set of at most $D$ vertex-disjoint 2-paths covering all but at most $\gamma n$ vertices of $H$.*

For non-extremal 4-graphs $H$, combining these three lemmas proves the reverse implication of Theorem 6, which we express in the following lemma.

▶ **Lemma 14.** *Suppose that $\frac{1}{n} \ll \varepsilon \ll c$ and that $n$ is even, and let $H$ be a 4-graph of order $n$ with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. If $H$ is neither $c$-odd-extremal nor $c$-even-extremal, then $H$ contains a Hamilton 2-cycle.*

**Proof sketch.** Introduce constants with $1/n \ll 1/D, \varepsilon \ll \gamma \ll \rho \ll \beta \ll \lambda \ll c, \mu \ll 1$, and apply Lemma 12 to obtain an absorbing 2-path $P_0$ in $H$ and a graph $G$ on $V(H)$ with the stated properties. Let $V := V(H)$ and $U := V(P_0)$, and now choose uniformly at random a set $R \subseteq V \setminus U$ of size $\rho n$. Next, apply Lemma 13 (with, say, $\eta = 1/10$) to obtain at most $D$

vertex-disjoint 2-paths $P_1, \dots, P_q$ in $H[V \setminus (U \cup R)]$ covering all but at most $\gamma n$ vertices. By $q$ applications of Lemma 11 we can find vertex-disjoint 2-paths $Q_0, Q_1, \dots, Q_q$, each of length at most 3, such that $Q_0$ connects the end of $P_0$ to the start of $P_1$, $Q_1$ connects the end of $P_1$ to the start of $P_2$, and so forth, with $Q_q$ connecting the end of $P_q$ to the start of $P_0$. Moreover, all vertices of $Q_i$ except those in the end of $P_i$ or the start of $P_{i+1}$ should be taken from $R$. (The random choice of $R$ ensures that the conditions of Lemma 11 are satisfied for each application.) This yields a 2-cycle $C = P_0Q_0P_1Q_1P_2 \dots P_qQ_q$ in $H$ covering all vertices except the at most $\gamma n$ vertices not covered by $P_1, \dots, P_q$ and between $\rho n - 3D$ and $\rho n$ unused vertices of $R$. So $X := V \setminus V(C)$ has size $\rho n - 3D \leq |X| \leq \rho n + \gamma n$. Furthermore, $|X|$ is even since $n$ and $|V(C)|$ are both even, and our random choice of $R$ ensures that every vertex $x \in X$ has $\deg_{G[X]}(x) \geq |X|/2$. So there is a perfect matching $e_1, \dots, e_{|X|/2}$ in $G[X]$; since $|X|/2 \leq \rho n$ we may 'absorb' $X$ into $P_0$ to obtain a 2-path $P^*$. Replacing $P_0$ by $P^*$ in $C$ gives a Hamilton 2-cycle in $H$. ◀

### 2.3.2 Extremal 4-Graphs

Having dealt with the 'non-extremal' case, it remains to deal with the two 'near-extremal' cases by proving the following two lemmas via an extremal case.

▶ **Lemma 15.** *Suppose that $\frac{1}{n} \ll \varepsilon, c \ll 1$ and that $n$ is even, and let $H$ be a 4-graph of order $n$ with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. If $H$ is $c$-even-extremal and every partition of $V(H)$ into two parts $A$ and $B$ is even-good, then $H$ contains a Hamilton 2-cycle.*

▶ **Lemma 16.** *Suppose that $\frac{1}{n} \ll \varepsilon, c \ll 1$ and that $n$ is even, and let $H$ be a 4-graph of order $n$ with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$. If $H$ is $c$-odd-extremal and every partition of $V(H)$ into two parts $A$ and $B$ is odd-good, then $H$ contains a Hamilton 2-cycle.*

**Proof sketch.** As is typical of this type of argument, each lemma is proved by a long and detailed extremal case analysis, and so we limit ourselves here to a brief outline of the argument for Lemma 15 (the outline for Lemma 16 is similar with 'even' and 'odd' reversed). Let $(A', B')$ be a partition of $V(H)$ witnessing that $H$ is $c$-even-extremal. We first observe that the bound on $\delta(H)$ implies that $H$ has density at least $(\frac{1}{2} - \varepsilon)$. Combined with the fact that $H$ has few odd edges, this implies that almost every set $S \subseteq V(H)$ for which $|A' \cap S|$ is even is an edge of $H$. However, it is possible that a small number of vertices may lie in very few even edges, so we begin by 'tidying up' the partition: we move a few vertices of $H$ from one side to the other to ensure that, for instance, every vertex of $H$ lies in many even edges. Let $(A, B)$ be the tidied partition. By assumption this partition $(A, B)$ is even-good, and this fact yields some structure in $H$ with respect to this partition (precisely what structure depends on the values of $n$ and $|A|$). For example, we might obtain two disjoint odd edges in $H$. We then form a short 2-path $P$ from the given structure to satisfy the desired parity conditions, and then (using even edges only) extend $P$ to a Hamilton 2-cycle in $H$. ◀

**Proof of Theorem 6.** Fix a constant $c$ small enough for Lemmas 15 and 16. Having done so, choose $\varepsilon$ sufficiently small for us to apply Lemma 14 with this choice of $c$, and $n_0$ sufficiently large that we may apply Lemmas 14, 15 and 16 with these choices of $c$ and $\varepsilon$ and any even $n \geq n_0$. Let $H$ be a 4-graph on $n$ vertices with $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$, and suppose that every partition $(A, B)$ of $V(H)$ is both even-good and odd-good. If $H$ is either $c$-even-extremal or $c$-odd-extremal then $H$ contains a Hamilton 2-cycle by Lemma 15 or 16 respectively. On the other hand, if $H$ is neither $c$-odd-extremal nor $c$-even-extremal then $H$ contains a Hamilton 2-cycle by Lemma 14. This completes the proof of the backwards implication of Theorem 6; the proof of the forwards implication was Proposition 9. ◀

### 2.4 Proof of Theorem 2

To conclude this section, we show how Theorem 2 can be deduced from Theorem 6. We begin by justifying the claim that the degree bound of Theorem 2 is best-possible. To see this, fix an even integer $n \geq 6$, and construct a 4-graph $H^*$ as follows. Let $A$ and $B$ be disjoint sets with $|A \cup B| = n$ such that $|A| = \frac{n}{2} - 1$ if 8 divides $n$ and $|A| = \frac{n}{2}$ otherwise. Then the vertex set of $H^*$ is $A \cup B$, and the edges of $H^*$ are all sets $e \in \binom{A \cup B}{4}$ such that $|e \cap A|$ is odd. Then it is easily checked that $\delta(H^*) = \frac{n}{2} - 3$ if 8 divides $n$ and $\frac{n}{2} - 2$ otherwise. Moreover, since $H^*$ has no even edges, our choice of size of $A$ implies that the partition $(A, B)$ of $V(H^*)$ is not odd-good. By Theorem 6 we conclude that there is no Hamilton 2-cycle in $H^*$.

**Proof of Theorem 2.** Choose $\varepsilon, n_0$ as in Theorem 6. Let $n \geq n_0$ be even and large enough that $\frac{n}{2} - 2 \geq (\frac{1}{2} - \varepsilon)n$, and let $H$ be a 4-graph on $n$ vertices which satisfies the minimum codegree condition of Theorem 2. Also let $(A, B)$ be a partition of $V(H)$, and assume without loss of generality that $|A| \leq \frac{n}{2}$. By Theorem 6 it suffices to prove that $(A, B)$ is even-good and odd-good. For this, note that if 8 divides $n$ and $|A| = \frac{n}{2}$ then $(A, B)$ is even-good by (i) and odd-good by (v). So we may assume that if 8 divides $n$ then $|A| \leq \frac{n}{2} - 1$ and $\delta(H) \geq \frac{n}{2} - 2$, whilst otherwise we have $|A| \leq \frac{n}{2}$ and $\delta(H) \geq \frac{n}{2} - 1$. Either way, we must have $\delta(H) \geq |A| - 1$. Also, for any distinct $x, y, z \in V(H)$, let $N_B(x, y, z)$ denote the set of vertices $w \in B$ such that $\{x, y, z, w\} \in E(H)$.

To see that $(A, B)$ must be even-good, arbitrarily choose vertices $x_1, x_2, y_1, y_2, z_1, z_2 \in A$. Then $|N_B(x_1, y_1, z_1)|, |N_B(x_2, y_2, z_2)| \geq \delta(H) - (|A| - 3) \geq 2$, so we may choose distinct $w_1, w_2 \in B$ with $w_1 \in N_B(x_1, y_1, z_1)$ and $w_2 \in N_B(x_2, y_2, z_2)$. The sets $\{x_1, y_1, z_1, w_1\}$ and $\{x_2, y_2, z_2, w_2\}$ are then disjoint odd edges of $H$, so $(A, B)$ is even-good by (ii).

We next show that $(A, B)$ is also odd-good. For this, arbitrarily choose distinct vertices $a_1, a_2, \dots, a_9, a'_1, \dots, a'_9 \in A$ and $b_1, \dots, b_9 \in B$. For any $1 \leq i, j \leq 9$ we have $|N_B(a_i, a'_i, b_j)| \geq \delta(H) - (|A| - 2) \geq 1$, so there must be $b^i_j \in B$ such that $\{a_i, a'_i, b_j, b^i_j\}$ is an (even) edge of $H$. If for each $1 \leq j \leq 9$ the vertices $b^i_j$ for $1 \leq i \leq 9$ are all distinct, then there is no set $X \subseteq V(H)$ with $|X| \leq 8$ which intersects every even edge of $H$. However, as observed immediately after Definition 5, such a set $X$ must exist if $(A, B)$ is not odd-good. We may therefore assume that $b^{i'}_j = b^i_j$ for some $1 \leq i, i', j \leq 9$ with $i \neq i'$. It follows that $\{a_i, a'_i, b_j, b^i_j\}$ is an even edge of $H$ with exactly two vertices in $A$, whilst $(a_i, a'_i, b_j, b^i_j, a_{i'}, a'_{i'})$ is a 2-path of length 2 in $H_{\text{even}}$. So $(A, B)$ is odd-good by (v), (vi), (vii) or (viii), according to the value of $n$ modulo 8. ◀

## 3 Tight Hamilton Cycles

Our aim in this section is to explain the principal ideas of the proof of Theorem 3, which proceeds by a series of reductions. We begin with a full proof of the case $k = 3$, in which case we proceed from a theorem of Garey, Johnson and Stockmeyer [5], who proved that the Hamilton cycle problem remains NP-complete when restricted to subcubic graphs (we say that a graph $G$ is *subcubic* if $G$ has maximum degree $\Delta(G) \leq 3$). The following proposition is an immediate corollary of that theorem.

▶ **Proposition 17** ([5]). *The problem of determining whether a subcubic graph admits a Hamilton path is NP-complete.*

The next lemma is the $k = 3$ case of Theorem 3, which holds with $C = 9$.

▶ **Lemma 18.** *The 3-graph tight Hamilton cycle decision problem is NP-complete even when restricted to 3-graphs $H$ on $m$ vertices with $\delta(H) \geq \frac{m}{2} - 9$.*

**Proof.** Let $G$ be a subcubic graph on $n$ vertices, and write $X := V(G)$. Assume for simplicity that $n$ is even (a very similar argument handles the case where $n$ is odd). Fix disjoint sets $A$ and $B$ with $|A| = \frac{3n}{2}$ and $|B| = \frac{3n}{2} + 1$ such that $X \subseteq A$, and define a 3-graph $H$ with vertex set $A \cup B$ whose edges are

**(i)** all sets $e \in \binom{A \cup B}{3}$ with $|A \cap e| \leq 1$,
**(ii)** all sets $e \in \binom{A \cup B}{3}$ with $|A \cap e| = 2$ and $A \cap e \in E(G)$ (note in particular that this requires that $A \cap e \subseteq X$), and
**(iii)** all sets $e \in \binom{A}{3}$ for which no $e' \in E(G)$ satisfies $e' \subseteq e$.

Observe first that $H$ has $m := 3n + 1$ vertices and minimum codegree $\delta(H) \geq \frac{m}{2} - 9$. To see this, let $x$ and $y$ be distinct vertices of $H$. If either $x \in B$ or $y \in B$ then $\{x, y, z\}$ is an edge of $H$ for any $z \in B \setminus \{x, y\}$, so $\deg_H(\{x, y\}) \geq |B| - 2 = \frac{3n}{2} - 1$. Exactly the same applies if $x, y \in A$ and $xy \in E(G)$. Finally, if $x, y \in A$ and $xy \notin E(G)$, then $\{x, y, z\}$ is an edge of $H$ for any $z \in A \setminus \{x, y\}$ except for those $z$ such that $xz \in E(G)$ or $yz \in E(G)$. So $\deg_H(\{x, y\}) \geq |A| - 2 - \deg_G(x) - \deg_G(y)$; since $G$ is subcubic this gives $\deg_H(\{x, y\}) \geq \frac{3n}{2} - 8 \geq \frac{m}{2} - 9$, as claimed.

We claim that $H$ contains a tight Hamilton cycle if and only if $G$ contains a Hamilton path. To see this, first suppose that $G$ contains a Hamilton path $(x_1, \cdots, x_n)$. Enumerate the vertices of $A \setminus X$ and $B$ as $a_1, a_2, \ldots, a_{n/2}$ and $b_1, b_2, \ldots, b_{3n/2+1}$ respectively. Then

$$\left(x_1, x_2, b_1, x_3, x_4, b_2, \cdots, x_{n-1}, x_n, b_{\frac{n}{2}}, b_{\frac{n}{2}+1}, a_1, b_{\frac{n}{2}+2}, b_{\frac{n}{2}+3}, a_2, \cdots, a_{\frac{n}{2}}, b_{\frac{3n}{2}}, b_{\frac{3n}{2}+1}\right)$$

is a tight Hamilton cycle in $H$.

Now suppose instead that $H$ contains a tight Hamilton cycle $C$. Note that our construction of $H$ ensures that there are no edges $e, e' \in E(H)$ with $|e \cap A| = 3$, $|e' \cap A| = 2$ and $|e \cap e'| = 2$. Since every edge of $C$ intersects the subsequent edge of $C$ in precisely two vertices, and $B \neq \emptyset$, it follows that $C$ cannot contain any edge $e$ with $|e \cap A| = 3$. So there are at least $\frac{n}{2}$ vertices $a \in X$ which are succeeded in $C$ by a vertex of $B$. Now let $A_1$ be the set of vertices of $X$ for which the subsequent vertex of $A$ on $C$ is in $X$ and $A_2$ be the set of vertices of $X$ for which the subsequent vertex of $A$ on $C$ is in $A \setminus X$. Also let $A_3 := A \setminus X$, so $A$ is the disjoint union of $A_1$, $A_2$ and $A_3$. By construction of $H$, any vertex of $A \setminus X$ must be preceded in $C$ by two vertices of $B$ and succeeded in $C$ by two vertices of $B$; it follows that any vertex of $A_2 \cup A_3$ is succeeded in $C$ by two vertices of $B$, and so we obtain

$$|B| \geq (\tfrac{n}{2} - |A_2|) + 2(|A_2| + |A_3|) = \tfrac{n}{2} + |A_2| + 2|A_3| = \tfrac{3n}{2} + |A_2|.$$

Since $A \setminus X$ is non-empty, we must have $|A_2| \geq 1$. Combined with the fact that $|B| = \frac{3n}{2} + 1$ this implies that $|A_2| = 1$, and all inequalities are in fact equalities. So precisely one vertex of $X$ is succeeded in $C$ by two vertices of $B$, $\frac{n}{2} - 1$ vertices of $X$ are succeeded by one vertex of $B$, and the remaining $\frac{n}{2}$ vertices of $X$ are succeeded by a vertex of $A$ (which must therefore be in $X$). This implies that $C$ contains a tight Hamilton path of the form $(x_1, x_2, b_1, x_3, x_4, b_2, \ldots, b_{n/2-1}, x_{n-1}, x_n)$, where $X = \{x_1, \ldots, x_n\}$ and $b_i \in B$ for $1 \leq i \leq \frac{n}{2} - 1$. By our construction of $H$ it follows that $(x_1, x_2, \ldots, x_n)$ is a Hamilton path in $G$.

Altogether, this shows that any instance of the Hamilton cycle problem for subcubic graphs can be reduced to a single instance of the problem of finding a tight Hamilton cycle in a 3-graph on $m$ vertices with $\delta(H) \geq \frac{m}{2} - 9$, where $m = 3n + 1$. Together with Proposition 17, this proves the lemma. ◀

We conclude by outlining the steps we use to prove Theorem 3 in full generality, using the following notation. For a function $f(n)$, we write $\mathrm{HC}(k, f(n))$ (respectively $\mathrm{HP}(k, f(n))$)

to denote the $k$-graph tight Hamilton cycle (respectively Hamilton path) decision problem restricted to $k$-graphs $H$ on $n$ vertices with minimum codegree $\delta(H) \geq f(n)$. On the other hand, for an integer $D$, we write $\overline{\mathrm{HC}}(k, D)$ (respectively $\overline{\mathrm{HP}}(k, D)$) to denote the $k$-graph tight Hamilton cycle (respectively Hamilton path) decision problem restricted to $k$-graphs $H$ with maximum codegree $\delta(H) \leq D$. So, for example, Proposition 17 states that $\overline{\mathrm{HP}}(2, 3)$ is NP-complete, whilst Lemma 18 states that $\mathrm{HC}(3, \frac{n}{2} - 9)$ is NP-complete. We prove Theorem 3 by exhibiting the following polynomial-time reductions.

**(i)** For any $k \geq 2$ and $D$ we give polynomial-time reductions from $\overline{\mathrm{HC}}(k, D)$ to $\overline{\mathrm{HP}}(k, D)$ and from $\overline{\mathrm{HP}}(k, D)$ and $\overline{\mathrm{HC}}(k, D)$. These reductions are elementary and permit us the convenience of treating the tight Hamilton cycle and tight Hamilton path problems in graphs of low maximum codegree as being interchangeable.

**(ii)** For any $k \geq 2$ we give polynomial-time reductions from $\overline{\mathrm{HC}}(k, D)$ to $\overline{\mathrm{HC}}(2k - 1, 2D)$ and from $\overline{\mathrm{HC}}(k, D)$ to $\overline{\mathrm{HC}}(2k, D)$. In each case, given a $k$-graph $H$ on a vertex set $V$, we take copies $H_1$ and $H_2$ of $H$ with disjoint vertex sets $V_1$ and $V_2$. For the former reduction we define a $(2k-1)$-graph $H^*$ on $V_1 \cup V_2$ whose edges are those $(2k-1)$-tuples which consist of an edge $e_1$ from $H_1$ and the copies in $H_2$ of $k - 1$ vertices of $e_1$, or the same with the roles of $H_1$ and $H_2$ reversed. Likewise, for the latter reduction we define a $2k$-graph $H^*$ on $V_1 \cup V_2$ whose edges are those $2k$-tuples $e_1 \cup e_2$ where $e_1$ is an edge of $H_1$, $e_2$ is an edge of $H_2$, and $e_2$ contains the copies of at least $k - 1$ vertices of $e_1$. In either case it is not too hard to show that $H^*$ contains a tight Hamilton cycle if and only if $H$ does, and that $\Delta(H^*) \leq 2\Delta(H)$ in one case and $\Delta(H^*) \leq \Delta(H)$ in the other.

**(iii)** Finally, for any $k \geq 2$ we present a polynomial-time reduction from $\overline{\mathrm{HP}}(k, D)$ to $\mathrm{HC}(2k - 1, \lfloor \frac{n}{2} \rfloor - k(D + 1))$ and from $\overline{\mathrm{HC}}(k, D)$ to $\mathrm{HC}(2k, \frac{n}{2} - k(D + 1))$. These are similar to the reduction given in the proof of Lemma 18, except that $G$ is now a $k$-graph with $\Delta(G) \leq D$, and $H$ is a $(2k - 1)$-graph or $2k$-graph (according to which reduction we are presenting).

By induction on $k$, with Proposition 17 as the base case, the reductions of (i) and (ii) combine to prove the following theorem, which can be seen as a generalisation to $k$-graphs of the aforementioned theorem of Garey, Johnson and Stockmeyer.

▶ **Theorem 19.** *For every $k \geq 2$ there exists $D$ such that $\overline{\mathrm{HC}}(k, D)$ and $\overline{\mathrm{HP}}(k, D)$ are NP-complete.*

Theorem 3 follows immediately from Theorem 19 and the reductions of (iii).

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